UWThPh–2007–18
HWM–07–28
EMPG–07–17

Nonabelian localization for gauge theory on the fuzzy sphere[1]

Harold Steinacker[1], Richard J. Szabo[2]
[1] Fakultät für Physik, Universität Wien
Boltzmanngasse 5, A-1090 Wien, Austria
[2] Department of Mathematics and
Maxwell Institute for Mathematical Sciences
Heriot-Watt University, Colin Maclaurin Building
Riccarton, Edinburgh EH14 4AS, U.K.

E-mail: harold.steinacker@univie.ac.at , R.J.Szabo@ma.hw.ac.uk

Abstract. We apply nonabelian equivariant localization techniques to Yang-Mills theory on the fuzzy sphere to write the partition function entirely as a sum over local contributions from critical points of the action. The contributions of the classical saddle-points are evaluated explicitly, and the partition function of ordinary Yang-Mills theory on the sphere is recovered in the commutative limit.

1. Introduction

The formulation of field theories on noncommutative spaces is expected to incorporate to some extent the effects of quantum gravity in a field theoretic framework (see e.g. [1, 2] for reviews, and [3, 4] concerning the relation with gravity). Their quantization, however, is rather non-trivial, due to a new phenomenon called UV/IR mixing. This problem appears to be very generic in noncommutative field theories, both for scalar and for gauge field theories. In essence it means that the ultraviolet divergences not only lead to the usual infinite renormalizations of the masses and couplings, but also to new divergences in the infrared behaviour of propagators, which are likely to signal new physics. It is therefore important to develop appropriate techniques for the quantization of noncommutative field theories, and to find models which are well-defined in order to avoid problems which are possibly associated to mathematical artifacts.

Fuzzy spaces provide a nice class of noncommutative spaces based on finite-dimensional algebras of "functions", with the same symmetries as their classical counterparts. This means that field theory on fuzzy spaces is naturally regularized, but the regularization is compatible with a geometrical symmetry group (in contrast to lattice field theory, for example). A large family of such spaces is given by the quantization of coadjoint orbits $\mathcal{O}$ of a Lie group in terms of certain finite matrix algebras $\mathcal{O}_N$. They are labelled by a noncommutativity parameter $\frac{1}{N}$,

[1] Based on talk given by H.S. at the International Conference "Noncommutative Geometry and Physics", April 23–27, 2007, Orsay, France. To be published in *Journal of Physics Conference Series*.

and the classical space is recovered in the large N limit. The simplest example is the fuzzy sphere S^2_N, which has been studied in great detail; see e.g. [5]–[9] and references therein. There are also extensively studied four-dimensional examples, such as $S^2_N \times S^2_N$ and $\mathbb{C}P^2_N$ [10]–[13].

In this article, we review the application of nonabelian localization techniques to Yang-Mills theory on S^2_N developed in [14]. This provides, along with [15], one of the few examples where noncommutative gauge theory can be solved exactly. We will explicitly evaluate the partition function and show that it reduces to the expected one on the classical sphere S^2 in the limit $N \to \infty$.

2. Equivariant Localization and the Duistermaat-Heckman theorem

Let X be a compact $2n$-dimensional symplectic manifold with symplectic two-form ω. Assume that the circle group $U(1)$ acts globally on X via symplectomorphisms, generated by a Hamiltonian vector field V with

$$\mathrm{d}H = -\iota_V \omega = -\omega(V,-) \tag{2.1}$$

for some real-valued function H on X. The Duistermaat-Heckman theorem (see e.g. [16] and references therein) then states that the classical partition function

$$Z = \int_X \frac{\omega^n}{n!}\ \mathrm{e}^{-\beta\, H} \tag{2.2}$$

is given *exactly* by the semi-classical approximation, i.e. by summing over all critical points P_i of H:

$$Z = \sum_i \frac{\mathrm{e}^{-\beta\, H(P_i)}}{\alpha_i}\ . \tag{2.3}$$

Here α_i is the product of the weights of the representation of the $U(1)$ action in the tangent space at P_i, which is formally given by the equivariant Euler class $e_V(P_i) = \mathrm{pfaff}\, \mathrm{d}V(P_i)$ of the normal bundle to the critical point set in X. As such, it is the fluctuation determinant determined by integration over an infinitesimal neighbourhood of P_i.

The subject of this paper is the application of a generalization of this theorem to compute the partition function of Yang-Mills theory on the fuzzy sphere. However, there are several complications which require a more sophisticated version of the localization formula. First, the global symmetry group $U(1)$ is replaced by the gauge group, which is nonabelian and usually infinite-dimensional; in the fuzzy case it becomes a finite-dimensional unitary group. Second, the saddle-points are replaced by critical surfaces. These complications can be handled using techniques from equivariant cohomology, following the method in [17] developed for ordinary two-dimensional Yang-Mills theory. In fact, the formal treatment in [17] is realized in our setting in a rigorous, finite-dimensional framework. We will also take advantage of some more recent techniques in [18] which allow for the explicit evaluation of the contributions from the classical solutions of the Yang-Mills equations of motion.

3. The fuzzy sphere

The fuzzy sphere S^2_N [5] is a matrix approximation of the usual sphere S^2. The algebra of functions on S^2, spanned by the spherical harmonics, is truncated at a given frequency. The algebra then becomes the finite-dimensional algebra of $N \times N$ matrices. More precisely, let $N \in \mathbb{N}$, and let ξ_i, $i = 1,2,3$ be the $N \times N$ hermitian coordinate generators of the fuzzy sphere $S^2_N \cong \mathrm{Mat}_N$ which satisfy the relations

$$\epsilon^{ij}{}_k\, \xi_i\, \xi_j = \mathrm{i}\, \xi_k \qquad \text{and} \qquad \xi_i\, \xi^i = \tfrac{1}{4}\, \left(N^2-1\right)\ \mathbb{1}_N \tag{3.1}$$

where throughout repeated upper and lower indices are implicitly summed over. The deformation parameter is $\frac{1}{N}$ and S^2_N becomes the algebra of functions on the classical unit sphere S^2 in the limit $N \to \infty$. The quantum space S^2_N preserves the classical invariance under global rotations as follows. The ξ_i generate an N-dimensional representation of the global $SU(2)$ isometry group. Under the adjoint action of $SU(2)$, this representation decomposes covariantly into p-dimensional irreducible representations (p) of $SU(2)$ as

$$\mathrm{Mat}_N \cong (1) \oplus (3) \oplus \cdots \oplus (2N-1) \ , \tag{3.2}$$

which are interpreted as fuzzy spherical harmonics. This decomposition defines a natural map from S^2_N to the space of functions on the commutative sphere. The integral of a function $f \in S^2_N$ over the fuzzy sphere is given by the trace of f, which coincides with the usual integral on S^2

$$\mathrm{Tr}(f) = \frac{N}{4\pi} \int_{S^2} \mathrm{d}\Omega \ f \tag{3.3}$$

where the above map is understood. Rotational invariance of the integral then corresponds to invariance of the matrix trace under the adjoint action of $SU(2)$.

Following [9], let us combine the generators ξ_i into a larger hermitian $\mathcal{N} \times \mathcal{N}$ matrix

$$\Xi = \tfrac{1}{2}\, \mathbb{1}_N \otimes \sigma^0 + \xi_i \otimes \sigma^i \tag{3.4}$$

where $\mathcal{N} = 2N$, $\sigma^0 = \mathbb{1}_2$, while

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5}$$

are the Pauli spin matrices obeying

$$\mathrm{Tr}\,(\sigma^i) = 0 \qquad \text{and} \qquad \sigma^i\,\sigma^j = \delta^{ij}\,\sigma^0 + \mathrm{i}\,\epsilon^{ij}{}_k\,\sigma^k \ . \tag{3.6}$$

One easily finds from (3.1) and (3.6) the identities

$$\Xi^2 = \tfrac{N^2}{4}\, \mathbb{1}_{\mathcal{N}} \qquad \text{and} \qquad \mathrm{Tr}(\Xi) = N \ . \tag{3.7}$$

Since $\xi_i \otimes \sigma^i$ is an intertwiner of the Clebsch-Gordan decomposition $(N) \otimes (2) = (N-1) \oplus (N+1)$, this implies that Ξ has eigenvalues $\pm \frac{N}{2}$ with respective multiplicities $N_\pm = N \pm 1$.

4. Gauge theory on the fuzzy sphere

4.1. Configuration space

We will now describe the gauge field degrees of freedom in our formulation. To elucidate the construction in as transparent a way as possible, we begin with the abelian case of $U(1)$ gauge theory. To introduce $\mathfrak{u}(1)$ gauge fields A_i on S^2_N, consider the covariant coordinates

$$C_i = \xi_i + A_i \qquad \text{and} \qquad C_0 = \tfrac{1}{2}\, \mathbb{1}_N + A_0 \tag{4.1}$$

which transform under the gauge group $U(N)$ as $C_\mu \mapsto U^{-1}\, C_\mu\, U$ for $\mu = 0, 1, 2, 3$ and $U \in U(N)$. We can again assemble them into a larger $\mathcal{N} \times \mathcal{N}$ matrix

$$C = C_\mu \otimes \sigma^\mu \ . \tag{4.2}$$

Generically these are four independent fields, and we have to somehow reduce them to two tangential fields on S^2_N. There are several ways to do this. For example, one can impose the

constraints $A_0 = 0$ and $C_i\, C^i = \frac{N^2-1}{4}\, \mathbb{1}_{\mathcal{N}}$ as in [9], leading to a constrained hermitian multi-matrix model describing quantum gauge theory on the fuzzy sphere which recovers Yang-Mills theory on the classical sphere in the large N limit.

Here we will use a different approach and impose the constraints

$$C^2 = \tfrac{N^2}{4}\, \mathbb{1}_{\mathcal{N}} \qquad \text{and} \qquad \mathrm{Tr}(C) = N \tag{4.3}$$

which is equivalent to requiring that C has eigenvalues $\pm\frac{N}{2}$ with multiplicities $N_\pm = N \pm 1$. In terms of the components of (4.2), this amounts to the constraints

$$C_i\, C^i + C_0^2 = \tfrac{N^2}{4}\, \mathbb{1}_{\mathcal{N}} \qquad \text{and} \qquad \mathrm{i}\, \epsilon_i{}^{jk}\, C_j\, C_k + \{C_0, C_i\} = 0\ . \tag{4.4}$$

We checked above that this is satisfied for $A_\mu = 0$, wherein $C = \Xi$. We can then consider the action of the unitary group $U(2N)$ given by

$$C \longmapsto U^{-1}\, C\, U \tag{4.5}$$

which generates a coadjoint orbit of $U(2N)$ and preserves the constraint (4.3). The gauge fields A_μ are in this way interpreted as fluctuations about the coordinates of the quantum space S^2_N. The constraint (4.3) ensures that the covariant coordinates (4.2) describe a dynamical fuzzy sphere. The gauge group $U(N)$ and the global isometry group $SU(2)$ of the sphere are subgroups of the larger symmetry group $U(2N)$. In particular, the generators of the gauge group are given by elements of the form $\phi = \phi_0 \otimes \sigma^0$.

We thus claim that a possible configuration space of gauge fields is given by the *single* coadjoint orbit

$$\mathcal{O} := \mathcal{O}(\Xi) = \{ C = U^{-1}\, \Xi\, U \mid U \in U(\mathcal{N}) \} \tag{4.6}$$

where $\Xi \in \mathfrak{u}(2N)$ is given by (3.4). Explicitly, dividing by the stabilizer of Ξ gives a representation of the orbit (4.6) as the symmetric space $\mathcal{O} \cong U(2N)/U(N+1) \times U(N-1)$ of dimension $\dim(\mathcal{O}) = 2(N^2-1)$. Therefore the orbit $\mathcal{O}$ captures the correct number of degrees of freedom at least in the commutative limit $N \to \infty$, where the gauge fields A_i become essentially tangent vector fields on S^2_N. This will be established in detail below. A similar construction was given in [12] for the case of $\mathbb{C}P^2$.

The tangent space to $\mathcal{O}(\Xi)$ at a point C is isomorphic to $T_C\mathcal{O} \cong \mathfrak{u}(\mathcal{N})/\mathfrak{r}$, where

$$\mathfrak{r} = \mathfrak{u}(N_+) \oplus \mathfrak{u}(N_-) \tag{4.7}$$

is the stabilizer subalgebra of Ξ. This identification is equivariant with respect to the natural adjoint action of the Lie group $U(\mathcal{N})$. Explicitly, tangent vectors to $\mathcal{O}(\Xi)$ at C have the form

$$V_\phi = \mathrm{i}\, [C, \phi] \tag{4.8}$$

for any hermitian element $\phi \in \mathfrak{u}(\mathcal{N})/\mathfrak{r}$,[2] which are just the generators of the unitary group $U(\mathcal{N})$ acting on $\mathcal{O}(\Xi)$ by the adjoint action. These actually describe vector fields on the entire orbit space $\mathcal{O}(\Xi)$. Here and in the following we use the symbol C to denote both elements of $\mathcal{O}(\Xi)$, as well as the matrix of overcomplete coordinate functions on $\mathcal{O}(\Xi)$ defined using the embeddings $\mathcal{O}(\Xi) \hookrightarrow \mathfrak{u}(\mathcal{N}) \hookrightarrow \mathbb{C}^{\mathcal{N}^2}$.

The generalization to nonabelian $U(n)$ gauge theory is very simple. One now takes

$$\mathcal{N} = 2n\, N \tag{4.9}$$

[2] With our conventions, the vector fields (4.8) are real.

and enlarges the matrix (3.4) to $\Xi \otimes \mathbb{1}_n$ (which we continue to denote as Ξ for ease of notation). The configuration space is given by the $U(\mathcal{N})$ orbit (4.6) with $C^2 = \frac{N^2}{4}\, \mathbb{1}_{\mathcal{N}}$ and

$$\mathrm{Tr}(C) = n\,N \ . \tag{4.10}$$

Then C has eigenvalues $\pm\,\frac{N}{2}$ of respective multiplicities $n\,(N \pm 1)$, and the configuration space

$$\mathcal{O} = U(2n\,N)/U(n\,N_+) \times U(n\,N_-) \tag{4.11}$$

describes $\mathfrak{u}(n)$-valued gauge fields on S^2_N. Its dimension is given by

$$\dim(\mathcal{O}) = 2n^2\, \left(N^2 - 1\right) \ . \tag{4.12}$$

4.2. Yang-Mills action

We claim that the action

$$S = S(C) := \tfrac{N}{g}\ \mathrm{Tr}\left(C_0 - \tfrac{1}{2}\ \mathbb{1}_{n\,N}\right)^2 \tag{4.13}$$

for $C \in \mathcal{O}$ reduces in the commutative limit $N \to \infty$ to the usual Yang-Mills action on the sphere S^2. It can therefore be taken as a definition of the Yang-Mills action on the fuzzy sphere S^2_N. We establish this explicitly below in the abelian case $n = 1$.

Consider the three-component field strength [9]

$$\begin{aligned} F_i &:= \mathrm{i}\,\epsilon_i{}^{jk}\, C_j\, C_k + C_i \\ &= \mathrm{i}\,\epsilon_i{}^{jk}\, [\xi_j, A_k] + \mathrm{i}\,\epsilon_i{}^{jk} A_j\, A_k + A_i \end{aligned} \tag{4.14}$$

where $C_i = \xi_i + A_i$ as in (4.1). To understand its significance, consider the "north pole" of S^2_N where $\xi_3 \approx \frac{N}{2}\, x_3 = \frac{N}{2}\ \mathbb{1}_N$ (with unit radius), and one can replace the operators

$$\mathrm{i}\ \mathrm{ad}_{\xi_i} \quad \longrightarrow \quad -\varepsilon_i{}^j\, \partial_j := -\varepsilon_{ij}\, \tfrac{\partial}{\partial x_j} \tag{4.15}$$

in the commutative limit for $i, j = 1, 2$. Hence upon identifying the classical gauge fields A_i^{cl} through

$$A_i^{\mathrm{cl}} = -\varepsilon_i{}^j\, A_j \ , \tag{4.16}$$

the "radial" component F_3 of the field strength (4.14) reduces in the commutative limit to the standard expression

$$F_3 \approx \partial_1 A_2^{\mathrm{cl}} - \partial_2 A_1^{\mathrm{cl}} + \mathrm{i}\,[A_1^{\mathrm{cl}}\,,\, A_2^{\mathrm{cl}}] \ . \tag{4.17}$$

The constraint (4.4) now implies

$$\begin{aligned} F_i + \{C_0 - \tfrac{1}{2}\ \mathbb{1}_N\,,\, C_i\} \ &= \ F_i + \{A_0\,,\, C_i\} \ &= \ 0\ , \\ \{\xi_i\,,\, A^i\} + A_0 + A_i\, A^i + A_0\, A_0 \ &= \ 0\ . \end{aligned} \tag{4.18}$$

Since only configurations with $A_0 = O(\frac{1}{N})$ have finite action (4.13) and ξ_3 is of order N, this implies that A_3, F_1 and F_2 are of order $\frac{1}{N}$ at the north pole, with A_1 and A_2 finite of order 1. In particular, only the radial component F_3 survives the $N \to \infty$ limit, with

$$F_3 = -\{A_0, C_3\} \approx -N\, A_0 \ . \tag{4.19}$$

This analysis can be made global by considering the "radial" field strength $F_r = x^i\, F_i$, which reduces to the usual field strength scalar on S^2. The action (4.13) thus indeed reduces to the usual Yang-Mills action in the commutative limit with dimensionless gauge coupling g, giving

$$S \approx \frac{1}{N\,g}\ \mathrm{Tr}(F_r)^2 \approx \frac{1}{4\pi\, g}\ \int_{S^2} \mathrm{d}\Omega\ (F_r)^2 \ . \tag{4.20}$$

4.3. Critical surfaces
The critical surfaces of the action (4.13) are easy to find. Since the most general variation of $C \in \mathcal{O}$ is given by $\delta C = [C, \phi]$, the critical points satisfy

$$0 = \mathrm{Tr}\left(\delta C_0 \left(C_0 - \tfrac{1}{2}\right)\right) = \mathrm{Tr}\left([C, \phi]\, C_0\right) = -\,\mathrm{Tr}\left(\phi\, [C, C_0]\right) \tag{4.21}$$

for arbitrary $\phi \in \mathfrak{u}(\mathcal{N}\,)$. Hence they are given by solutions of the equation

$$[C_0, C] = 0 \ . \tag{4.22}$$

This agrees with the known saddle-points in the formulation of [9]. The equation $[C_0, C_i] = 0$ together with $C^2 = \frac{N^2}{4}\, 1\!\!1_{\mathcal{N}}$ in (4.4) implies that

$$\begin{aligned} [C_i, C_j] &= \mathrm{i}\, \epsilon_{ijk}\, (2C_0)\, C_k \ , \\ C_0^2 &= \frac{N^2}{4} - \sum_{i=1}^{3} C_i^2 \ . \end{aligned} \tag{4.23}$$

This means that C_i generates an $SU(2)$-module $\pi_{n\,N}$ given by a sum of irreducible representations of (4.23) characterized by partitions $\vec{n} = (n_1, \ldots, n_k)$ of the integer

$$n\,N = n_1 + \ldots + n_k \ , \tag{4.24}$$

where $n_i \in \mathbb{N}$ is the dimension of the i-th irreducible subrepresentation in the representation $\pi_{n\,N}$. Therefore each critical point is labelled (up to gauge transformations) by the set of dimensions n_i of the irreducible representations, supplemented by a "sign" which is defined by $s_i = \mathrm{sgn}(C_0(n_i)) = \pm 1$ (in that irreducible representation) if $C_0(n_i) \neq 0$, and $s_i = 0$ if $C_0(n_i) = 0$. We denote the collection of them by $\mathcal{C}_{(n_1,s_1),\ldots,(n_k,s_k)}$.

In particular, the "classical" saddle-points which in the commutative limit $N \to \infty$ go over to the saddle-points of classical Yang-Mills theory on S^2 (often called instantons) are given by the critical surfaces $\mathcal{C}_{(n_1,1),\ldots,(n_n,1)}$ with

$$n_i = N - m_i \qquad \text{and} \qquad \sum_{i=1}^{n} m_i = 0 \tag{4.25}$$

with small $m_i \in \mathbb{Z}$, for which

$$C_0(n_i) = \tfrac{N}{2(N-m_i)}\ 1\!\!1_{n_i} \approx \tfrac{1}{2}\left(1 + \tfrac{m_i}{N}\right)\, 1\!\!1_{n_i} \ . \tag{4.26}$$

Note that then

$$\mathrm{Tr}(C_0) = \frac{n\,N}{2} \tag{4.27}$$

as required. It follows that the action (4.13) evaluated on these solutions is given by

$$S((n_1, 1)\,, \ \ldots \ , \,(n_n, 1)) \approx \frac{1}{4g} \sum_{i=1}^{n} m_i^2 \ , \tag{4.28}$$

which is the usual expression [19, 20] for the classical action of $U(n)$ Yang-Mills theory on the sphere S^2 with trivial gauge bundle evaluated on the two-dimensional instanton on S^2 corresponding to a configuration of n Dirac monopoles of magnetic charges $m_i \in \mathbb{Z}$. Non-trivial gauge bundles over S^2 of first Chern class $\mathsf{c}_1 \in \mathbb{Z}$ are obtained by modifying the trace constraint as in [14]. All other non-classical saddle-points such as fluxons are suppressed at least by factors $\mathrm{e}^{\,-N\,g}$, reflecting the fact that their action becomes infinite in the commutative limit $N \to \infty$.

4.4. Partition function

We can now proceed to compute the partition function of quantum Yang-Mills theory on the fuzzy sphere defined by the action (4.13) on the configuration space (4.6) of gauge fields. The crucial aspect of the above formulation of $U(n)$ Yang-Mills theory on S_N^2 is that the space of gauge fields $\mathcal{O}$ in (4.6) or (4.11) is a coadjoint orbit. This implies that it is in particular a symplectic (and even Kähler) space with symplectic two-form ω, which is given explicitly by the usual Kirillov-Kostant construction

$$\langle\omega, V_\phi\wedge V_\psi\rangle = \mathrm{i}\ \mathrm{Tr}\left(C\,[\phi,\psi]\right) \tag{4.29}$$

where V_ϕ, V_ψ are tangent vectors to $\mathcal{O}$ as in (4.8).

After an irrelevant shift of the covariant coordinates (4.1) which is equivalent to working with the reduced Yang-Mills action

$$S' = S + \frac{n\,N^2}{4g}\ , \tag{4.30}$$

the partition function is defined by

$$\begin{aligned} Z' &:= \frac{1}{\mathrm{vol}(G)}\left(\frac{g}{4\pi\,N}\right)^{\dim(G)/2}\int_{\mathcal{O}}\mathrm{d}C\ \exp\left(-\tfrac{N}{g}\ \mathrm{Tr}\left(C_0^2\right)\right) \\ &= \frac{1}{\mathrm{vol}(G)}\left(\frac{g'}{2\pi}\right)^{\dim(G)/2}\int_{\mathcal{O}}\exp\left(\omega-\tfrac{1}{2g'}\ \mathrm{Tr}\left(C_0^2\right)\right) \end{aligned} \tag{4.31}$$

where we have used the fact that the symplectic volume form $\omega^d/d!$, with $d := \dim_{\mathbb{C}}(\mathcal{O})$, defines the natural gauge invariant measure on $\mathcal{O}$ provided by the Cartan-Killing riemannian volume form (up to some irrelevant normalization). This follows from the fact that the natural invariant metric on $\mathcal{O}$ is a Kähler form. We have divided by the volume of the gauge group $G = U(n\,N)$ with respect to the invariant Cartan-Killing form and by another normalization factor for later convenience, and also introduced the rescaled gauge coupling

$$g' = \frac{g}{2N}\ . \tag{4.32}$$

We will now describe, following [17, 15], how the technique of nonabelian localization can be applied to evaluate the symplectic integral (4.31) exactly.

We begin by using a gaussian integration to rewrite (4.31) as

$$Z' = \frac{1}{\mathrm{vol}(G)}\int_{\mathfrak{g}\times\mathcal{O}}\left[\frac{\mathrm{d}\phi}{2\pi}\right]\ \exp\left(\omega-\mathrm{i}\ \mathrm{Tr}(C_0\,\phi)-\tfrac{g'}{2}\ \mathrm{Tr}\left(\phi^2\right)\right)\ , \tag{4.33}$$

where the euclidean measure for integration over the gauge algebra $\phi\in\mathfrak{g}=\mathfrak{u}(n\,N)$ is determined by the invariant Cartan-Killing form. It is not hard to show that $H_\phi = \mathrm{Tr}(C_0\,\phi)$ is the moment map for the action of the gauge group, which means that

$$\mathrm{d}\,\mathrm{Tr}(C_0\,\phi) = -\iota_{V_\phi}\omega\ . \tag{4.34}$$

Introduce the BRST operator

$$Q = \mathrm{d} - \mathrm{i}\,\iota_{V_\phi}\ , \tag{4.35}$$

where d is the exterior derivative on $\Omega(\mathcal{O})$ and the contraction ι_{V_ϕ} acts trivially on ϕ. It preserves the gradation if one assigns charge $+2$ to the elements ϕ of $\mathfrak{g}$, and it satisfies

$$Q^2 = -\,\mathrm{i}\,\{\mathrm{d},\iota_{V_\phi}\} = -\,\mathrm{i}\,\mathcal{L}_{V_\phi} \tag{4.36}$$

where $\mathcal{L}_{V_\phi}$ is the Lie derivative along the vector field V_ϕ. Thus $Q^2 = 0$ exactly on the space

$$\Omega_G(\mathcal{O}) := \big(\mathbb{C}[[\mathfrak{g}]] \otimes \Omega(\mathcal{O})\big)^G \tag{4.37}$$

consisting of gauge invariant differential forms on $\mathcal{O}$ which take values in the ring of symmetric functions on the Lie algebra $\mathfrak{g}$.

By construction one has

$$Q(\omega - \mathrm{i}\ \mathrm{Tr}(C_0\,\phi)) = 0 \tag{4.38}$$

using $\mathrm{d}\omega = 0$ and (4.34), and

$$Q\,\mathrm{Tr}\,(\phi^2) = 0\ . \tag{4.39}$$

Therefore, the integrand of the partition function (4.33) defines a G-equivariant cohomology class in $H_G(\mathcal{O})$, and the value of Z' depends only on this class. The integral of any Q-exact equivariant differential form in $\Omega_G(\mathcal{O})$ over $\mathfrak{g}\times\mathcal{O}$ is clearly 0, as is the integral of any ι_{V_ϕ}-exact form even if its argument is not gauge invariant. Thus Z' is unchanged by adding any Q-exact form to the action, which will fix a gauge for the localization. Hence we can replace it by

$$Z' = \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{O}} \left[\frac{\mathrm{d}\phi}{2\pi}\right]\ \exp\Big(\omega - \mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \tfrac{g'}{2}\ \mathrm{Tr}\,(\phi^2) + t\ Q\alpha\Big)\ , \tag{4.40}$$

which is independent of $t\in\mathbb{R}$ for any G-invariant one-form α on $\mathcal{O}$, where

$$Q\alpha = \mathrm{d}\alpha - \mathrm{i}\,\langle\alpha, V_\phi\rangle\ . \tag{4.41}$$

The independence of (4.40) on the particular representative $\alpha\in\Omega(\mathcal{O})^G$ of its equivariant cohomology class will play a crucial role in our evaluation of the partition function.

Expanding the integrand of (4.40) by writing $\exp(t\ \mathrm{d}\alpha)$ as a polynomial in t and using the fact that the configuration space $\mathcal{O}$ is compact, it follows that for $t\to\infty$ the integral localizes at the stationary points of $\langle\alpha, V_\phi\rangle$ in $\mathfrak{g}\times\mathcal{O}$. By writing $V_\phi = V_a\ \phi^a$, where ϕ^a is an orthonormal basis of $\mathfrak{g}^\vee$, we have $\langle\alpha, V_\phi\rangle = \langle\alpha, V_a\rangle\ \phi^a$ and the critical points are thus determined by the equations

$$\langle\alpha, V_a\rangle = 0\ , \tag{4.42}$$

$$\phi^a\ \mathrm{d}\langle\alpha, V_a\rangle = 0\ . \tag{4.43}$$

Since (4.43) is invariant under rescaling of ϕ and the Lie algebra $\mathfrak{g}$ is contractible, the homotopy type of the space of solutions in $\mathfrak{g}\times\mathcal{O}$ is unchanged by restricting to $\phi = 0$ and the saddle-points reduce to the zeroes of $\langle\alpha, V_a\rangle$ in $\mathcal{O}$.

Let us consider explicitly the invariant one-form α given by [17, 15]

$$\alpha = -\,\mathrm{i}\ \mathrm{Tr}\,(C_0\,[C, \mathrm{d}C]_0)\ . \tag{4.44}$$

We claim that the vanishing locus of $\langle\alpha, V_a\rangle$ in this case coincides with the critical surfaces of the original Yang-Mills action (4.13) as found in Section 4.3. To see this, we note that the condition

$$0 = \langle\alpha, V_a\rangle = \mathrm{Tr}\,(C_0\,[C\ ,\ [C, \phi^a]\,]_0) = -\,\mathrm{Tr}\,([C, C_0]\,[C, \phi^a]) \tag{4.45}$$

certainly holds whenever $[C, C_0] = 0$. On the other hand, by setting $\phi = C_0$ it implies

$$0 = \langle\alpha, V_\phi\rangle = -\,\mathrm{Tr}\,([C, C_0]^2) \tag{4.46}$$

which by nondegeneracy of the inner product defined by the trace implies that $[C, C_0] = 0$. Therefore the action in (4.40) has indeed the same critical points as the Yang-Mills action (4.13).

Let us now explicitly establish, following [15], the localization of the partition function onto the classical solutions of the gauge theory. Plugging (4.44) and (4.41) into (4.40) and carrying out the integration over $\phi \in \mathfrak{g}$ gives

$$\begin{aligned} Z' &= \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{O}} \left[\frac{\mathrm{d}\phi}{2\pi}\right] \exp\Big(t\ \mathrm{d}\alpha + \omega\Big) \\ &\qquad \times\ \exp\Big(-\ \mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \tfrac{g'}{2}\,\mathrm{Tr}\,(\phi^2) - \mathrm{i}\,t\ \mathrm{Tr}\,([C\ ,\ [C, C_0]\,]\,\phi)\Big) \\ &= \frac{1}{\mathrm{vol}(G)} \left(\frac{g'}{2\pi}\right)^{\dim(G)/2} \int_{\mathcal{O}} \exp\Big(t\ \mathrm{d}\alpha + \omega\Big) \\ &\qquad \times\ \exp\Big(-\ \tfrac{1}{2g'}\ \mathrm{Tr}\,(C_0^2) + \tfrac{t}{g'}\ \mathrm{Tr}\,(C_0\,[C\ ,\ [C, C_0]\,]) - \tfrac{t^2}{2g'}\ \mathrm{Tr}\,([C\ ,\ [C, C_0]\,])^2\Big) \end{aligned} \tag{4.47}$$

where we have used $\mathrm{Tr}(C\,[C, -]) = 0$. The only configurations which contribute to (4.47) in the large t limit are therefore solutions of the equation

$$[C\ ,\ [C, C_0]\,] = 0 \tag{4.48}$$

which implies as in [15] that

$$0 = \mathrm{Tr}\,(C_0\,[C\ ,\ [C, C_0]\,]) = -\ \mathrm{Tr}\,([C, C_0]^2)\ , \tag{4.49}$$

giving $[C, C_0] = 0$ as desired. Therefore the integral (4.47) receives contributions only from the solutions of the Yang-Mills equations (4.22), which establishes the claimed localization.

The local geometry in $\mathfrak{g} \times \mathcal{O}$ about each critical point determines the partition function as a sum of local contributions involving the values of the Yang-Mills action evaluated on the classical solutions. This is gotten by considering an equivariant tubular neighbourhood $\mathcal{N}_{(n_1,s_1),\dots,(n_k,s_k)}$ of each critical surface $\mathcal{C}_{(n_1,s_1),\dots,(n_k,s_k)}$ in $\mathfrak{g}\times\mathcal{O}$. Since the partition function (4.40) is independent of t, we can consider its large t limit as above, and this limit will always be implicitly assumed from now on. Let $\mathcal{W}$ be a compact subset of $\mathcal{O}$ with $\mathcal{W} \cap \mathcal{C} = \emptyset$, where $\mathcal{C} := \bigcup_{(n_i,s_i)} \mathcal{C}_{(n_1,s_1),\dots,(n_k,s_k)}$. Then the integral over $\mathcal{W}$ in (4.47) has a gaussian decay in $t \to \infty$. This means that in expanding $\exp(t\ \mathrm{d}\alpha + \omega)$ into a finite sum of terms of the form $\omega^p \wedge (t\ \mathrm{d}\alpha)^m$, we can disregard all terms which contain ω since they will be suppressed by factors of $\frac{1}{t}$ and vanish in the large t limit. The only terms which survive the $t \to \infty$ limit are those with $p = 0, m = d$, and the integral therefore vanishes unless ω is replaced by $\mathrm{d}\alpha$, except at the saddle point where $\mathrm{d}\alpha = 0$. Then one has

$$Z' = \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{O}} \left[\frac{\mathrm{d}\phi}{2\pi}\right] \exp\Big(t\,(\mathrm{d}\alpha - \mathrm{i}\,\langle\alpha, V_\phi\rangle)\Big)\ \exp\Big(-\ \mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \tfrac{g'}{2}\ \mathrm{Tr}\,(\phi^2)\Big) \tag{4.50}$$

in the vicinity of any critical point in which $\mathrm{d}\alpha$ is nondegenerate.

5. Local geometry of the configuration space

To proceed with the explicit evaluation of the contributions from each critical surface $\mathcal{C}_{(n_1,1),\dots,(n_k,1)}$ to the partition function (4.50) for gauge theory on the fuzzy sphere, we need to describe the local geometry of the configuration space $\mathcal{O}$ in the infinitesimal neighbourhoods $\mathcal{N}_{(n_1,s_1),\dots,(n_k,s_k)}$. This is achieved using the explicit form of the complex structure $\mathcal{J}$ on the coadjoint orbits, together with equivariance under global $SU(2)$ rotations.

5.1. Complex structure

Consider for fixed $C \in \mathcal{O}$ the map

$$\begin{aligned} \mathcal{J} \, : \, \mathfrak{u}(\mathcal{N}\,) \quad &\longrightarrow \quad \mathfrak{su}(\mathcal{N}\,) \\ \phi \quad &\longmapsto \quad \tfrac{1}{N}\, V_\phi = \tfrac{\mathrm{i}}{N}\,[C\,,\,\phi] \end{aligned} \tag{5.1}$$

where as always the tangent space $T_C\mathcal{O}$ at C is viewed as a subspace of the ambient space $\mathfrak{u}(\mathcal{N}\,) \supset \mathcal{O}$. It is easy to check that it satisfies

$$\mathcal{J}^3 = -\mathcal{J} \ . \tag{5.2}$$

The map $\mathcal{J}$ is the complex structure on $T_C\mathcal{O} = \mathrm{im}(\mathcal{J})$. It provides the Cartan decomposition corresponding to the symmetric space $\mathcal{O}$:

$$\mathfrak{u}(\mathcal{N}\,) \ = \ \ker(\mathcal{J}) \ \oplus \ \underbrace{\ker\left(\mathcal{J}^2 + 1\!\!1_{\mathcal{N}}\right)}_{T_C\mathcal{O}} \ . \tag{5.3}$$

Here $\mathfrak{r} = \ker(\mathcal{J}) = \mathfrak{u}(n\,N_+) \oplus \mathfrak{u}(n\,N_-)$ is the stabilizer subalgebra of the coadjoint orbit $\mathcal{O}$.

Now consider the map

$$\begin{array}{ccccc} \mathfrak{g} & \longrightarrow & \mathcal{J}(\mathfrak{g}) & \longrightarrow & \mathcal{J}^2(\mathfrak{g}) \\ & & \text{(pure gauge)} & & \text{(physical)} \end{array} \tag{5.4}$$

which defines subspaces $\mathcal{J}(\mathfrak{g}), \mathcal{J}^2(\mathfrak{g})$ of $T_C\mathcal{O}$. We denote with $\mathfrak{s} \subset \mathfrak{g}$ the stabilizer of the gauge transformations, with $\mathcal{J}(\mathfrak{s}) = 0$. One can show [14] that for the vacuum solution $C = \Xi = \frac{1}{2}\ 1\!\!1_{\mathcal{N}} + \xi_i \otimes 1\!\!1_n \otimes \sigma^i$, there is a splitting

$$T_C\mathcal{O} = \mathcal{J}(\mathfrak{g}) \oplus \mathcal{J}^2(\mathfrak{g}) \tag{5.5}$$

while the structure of the generic critical surfaces is somewhat more complicated:

$$\mathcal{J}(\mathfrak{g} \ominus \mathfrak{h}) \oplus \mathcal{J}^2(\mathfrak{g} \ominus \mathfrak{h}) \oplus E_0 \oplus E_1 \ = \ T_C\mathcal{O} \tag{5.6}$$

where the subalgebra $\mathfrak{h}$ is defined via

$$E_0 = \mathcal{J}(\mathfrak{g}) \cap \mathcal{J}^2(\mathfrak{g}) = \mathcal{J}(\mathfrak{h}) = \mathcal{J}^2(\mathfrak{h}) \tag{5.7}$$

and E_1 is an extra vector space. To determine the vector spaces E_0, E_1 explicitly, we need to describe the decomposition under the global $SU(2)$ symmetry. We will only sketch the resulting structure and refer the interested reader to [14] for details.

5.2. $SU(2)$-equivariant decomposition at critical surfaces

The critical surface $\mathcal{C}_{(n_1,1),\dots,(n_n,1)}$ defines $SU(2)$ generators

$$J_i = \frac{C_i}{2C_0} + \frac{1}{2}\,\sigma_i \qquad \text{with} \quad [J_i, C] = 0 \ . \tag{5.8}$$

We decompose everything under this action of $SU(2)$. For the $n\,N$-dimensional representation $\pi_{n\,N} : \mathfrak{su}(2) \to \mathrm{End}(V)$ one has

$$V \otimes \mathbb{C}^2 = \Big(\bigoplus_{i=1}^{n} (n_i + 1) \Big) \ \oplus \ \Big(\bigoplus_{i=1}^{n} (n_i - 1) \Big) \tag{5.9}$$

so that

$$C=\frac{N}{2}\begin{pmatrix}\bigoplus\limits_{i=1}^{n}\mathbb{1}_{(n_i+1)} & 0\\ 0 & -\bigoplus\limits_{i=1}^{n}\mathbb{1}_{(n_i-1)}\end{pmatrix}\ \subset\ \mathfrak{u}(\mathcal{N}) \tag{5.10}$$

and

$$T_C\mathcal{O}\cong\left\{\begin{pmatrix}0 & X\\ X^\dagger & 0\end{pmatrix}\ \Big|\ X\in\mathrm{Mat}_{n\,N}\right\}\ \subset\ \mathfrak{u}(\mathcal{N})\ . \tag{5.11}$$

It follows that

$$T_C\mathcal{O}\cong\bigoplus_{i,j=1}^{n}(n_i+1)\otimes(n_j-1) \tag{5.12}$$

and

$$\mathfrak{g}\cong\bigoplus_{i,j=1}^{n}(n_i)\otimes(n_j)\ . \tag{5.13}$$

All this allows for the explicit computation of $\mathcal{J},E_0,E_1$ for the various critical surfaces $\mathcal{C}_{(n_1,1),\dots,(n_n,1)}$ as follows:

1) *Vacuum surface* $\mathcal{C}_{(N,1),\dots,(N,1)}$
The vacuum surface is the orbit through $C=\Xi=\frac{1}{2}\,\mathbb{1}_{\mathcal{N}}+\xi_i\otimes\mathbb{1}_n\otimes\sigma^i$. The stabilizer is given by $\mathfrak{s}=\mathfrak{u}(n)\subset\mathfrak{g}$, and

$$\mathfrak{g}\cong\big((1)\ \oplus\ (N+1)\otimes(N-1)\big)\otimes\mathfrak{u}(n) \tag{5.14}$$

gives

$$T_C\mathcal{O}=\mathcal{J}(\mathfrak{g})\oplus\mathcal{J}^2(\mathfrak{g}) \tag{5.15}$$

as in (5.5), which can be seen even by just counting dimensions.

2) *Maximally non-degenerate critical surface* $\mathcal{C}_{(n_1,1),\dots,(n_n,1)}$
For a generic critical surface $\mathcal{C}_{(n_1,1),\dots,(n_n,1)}$ with $n_1>n_2>\dots>n_n$ one finds

$$\mathfrak{g}\cong\bigoplus_{i,j=1}^{n}(n_i)\otimes(n_j)=\bigoplus_{i,j=1}^{n}\big((|n_i-n_j|+1)\oplus\cdots\oplus(n_i+n_j-1)\big)\ . \tag{5.16}$$

This gives

$$T_C\mathcal{O}=\mathcal{J}(\mathfrak{g}\ominus\mathfrak{h})\oplus\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{h})\oplus E_0\oplus E_1 \tag{5.17}$$

as in (5.6) with

$$E_1=\bigoplus_{i>j}(|n_i-n_j|-1)\qquad\text{and}\qquad E_0=\bigoplus_{i>j}(|n_i-n_j|+1)\ . \tag{5.18}$$

6. Nonabelian localization at the vacuum surface

We will first consider the localization of the partition function (4.50) at the vacuum orbit

$$\mathcal{O}_0:=\mathcal{C}_{(N,1),\dots,(N,1)}=\{g\,\Xi\,g^{-1}\ |\ g\in G\}\cong G/U(n)\ \subset\ \mathcal{O} \tag{6.1}$$

with gauge group $G=U(n\,N)$ and stabilizer $\mathfrak{s}=\mathfrak{u}(n)$.

6.1. Statement of result

Theorem 1. *The contribution to the quantum partition function for $U(n)$ Yang-Mills theory on S^2_N from the vacuum moduli space $\mathcal{O}_0$ is given by*

$$Z_0 = \frac{1}{n!}\,\frac{1}{(2\pi)^{n^2+n}}\,\int_{\mathbb{R}^n}[\mathrm{d}s]\ \Delta(s)^2\ \mathrm{e}^{-\frac{g}{4}\sum_i s_i^2}\ . \tag{6.2}$$

Here

$$\Delta(s) = \prod_{i<j}(s_i - s_j) = \det_{1\leq i,j\leq n}\big(s_i^{j-1}\big) \tag{6.3}$$

is the Vandermonde determinant, and we substituted back the original Yang-Mills action S using the shift (4.30). The quantum fluctuation integral (6.2) is the standard expression [19] for the contribution from the global minimum of the Yang-Mills action on S^2 to the $U(n)$ sphere partition function. It arises from the trivial instanton configuration with vanishing monopole charges $m_i = 0$ in (4.25).

6.2. Proof of Theorem 1

Localization implies that we can restrict ourselves to a G-equivariant tubular neighbourhood $\mathcal{N}_0 = \mathcal{N}_{(N,1),\dots,(N,1)}$ of the critical surface, under the action of the gauge group $G = U(n\,N)$. The neighbourhood $\mathcal{N}_0$ has an equivariant retraction by a local equivariant symplectomorphism onto the *local symplectic model* $\mathcal{F}_0$. This means that the tangent space to $\mathcal{F}_0$ at the vacuum critical point C is given by $T_C\mathcal{O}_0 \oplus \mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{s}) \cong \mathcal{J}(\mathfrak{g}\ominus\mathfrak{s})\oplus\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{s}) = T_C\mathcal{O}$, and the symplectic two-form on $\mathcal{F}_0$ is simply ω. In physical terms, the gauge fields are decomposed along the vacuum moduli space $\mathcal{O}_0$ plus infinitesimal non-gauge variations in the subspace $\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{s})$.

We need to introduce an explicit basis of $T_C\mathcal{O}$ and of the dual space of one-forms $\Omega^1(\mathcal{O})$. According to (5.15), we take the basis of vector fields to be

$$J_i = \mathcal{J}(g_i')\ ,\quad \tilde{J}_j = \mathcal{J}^2(g_j')\ \in\ T_C(\mathcal{O}) \tag{6.4}$$

where g_i' is an orthonormal basis of $\mathfrak{g}\ominus\mathfrak{s}$. The dual basis of one-forms

$$\lambda^i\ ,\quad \tilde{\lambda}^j\ \in\ \Omega^1(\mathcal{O}) \tag{6.5}$$

satisfy

$$\langle\lambda^i\,,\,J_j\rangle = \delta^i{}_j\ ,\quad \langle\tilde{\lambda}^i\,,\,\tilde{J}_j\rangle = \delta^i{}_j \qquad\text{and}\qquad \langle\lambda^i\,,\,\tilde{J}_j\rangle = \langle\tilde{\lambda}^i\,,\,J_j\rangle = 0\ . \tag{6.6}$$

Now introduce functions $f_i = \langle\alpha, J_i\rangle$. One can show that $\langle\alpha,\mathcal{J}^2(\mathfrak{g})\rangle = 0$, which implies that the localization one-form can be expanded as

$$\alpha = f_i\ \lambda^i \tag{6.7}$$

with

$$\mathrm{d}\alpha = \mathrm{d}f_i\wedge\lambda^i + f_i\ \mathrm{d}\lambda^i\ . \tag{6.8}$$

In particular, one has

$$\frac{(\mathrm{d}\alpha)^d}{d!} = \bigwedge_{i=1}^{d}\big(\mathrm{d}f_i\wedge\lambda^i\big) \tag{6.9}$$

up to forms which vanish on-shell, and hence are killed by localization in the large t limit. Here $d = \dim_{\mathbb{C}}(\mathcal{O}) = n^2\,(N^2-1)$ is the (real) dimension of the vacuum orbit $\mathcal{O}_0$.

We can now proceed with the evaluation of the local contribution to the partition function (4.50) for $t \to \infty$:

$$\begin{aligned} Z_0' &= \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{F}_0} \left[\frac{\mathrm{d}\phi}{2\pi}\right] \frac{t^d}{d!}\,(\mathrm{d}\alpha)^d\ \mathrm{e}^{-\,\mathrm{i}\,t\,\langle\alpha,V_\phi\rangle-\mathrm{i}\ \mathrm{Tr}(C_0\,\phi)-\frac{g'}{2}\ \mathrm{Tr}(\phi^2)} \\ &= \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{F}_0} \left[\frac{\mathrm{d}\phi}{2\pi}\right] t^d\ \bigwedge_{i=1}^{d} \left(\mathrm{d}f_i\wedge\lambda^i\right)\ \mathrm{e}^{-\,\mathrm{i}\,N\,t\,f_i\,\phi^i-\mathrm{i}\ \mathrm{Tr}(C_0\,\phi)-\frac{g'}{2}\ \mathrm{Tr}(\phi^2)} \\ &= \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{s}} \left[\frac{\mathrm{d}\phi}{2\pi}\right]\ \mathrm{e}^{-\,\mathrm{i}\ \mathrm{Tr}(C_0\,\phi)-\frac{g'}{2}\ \mathrm{Tr}(\phi^2)}\ \frac{1}{N^d} \int_{\mathcal{O}_0} \bigwedge_{i=1}^{d} \lambda^i\ . \end{aligned} \tag{6.10}$$

Here the f_i integrals over the fibre $\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{s})$ have produced delta-functions setting $\phi^i=0$ in $\mathfrak{g}\ominus\mathfrak{s}$. We can carry out the integral over the moduli space $\mathcal{O}_0$ in (6.10) by observing that

$$\frac{1}{N^d}\int_{\mathcal{O}_0} \bigwedge_{i=1}^{d}\lambda^i = \int_{G/S} \bigwedge_{i=1}^{d}\eta^i = \frac{\mathrm{vol}(G)}{\mathrm{vol}(S)}\ , \tag{6.11}$$

where the pullbacks $\mathcal{J}^*(\lambda^i)=\eta^i$ define left-invariant one-forms on the gauge group G.

To evaluate the remaining integral over the gauge stabilizer algebra $\mathfrak{s}\cong\mathfrak{u}(n)$ in (6.10), we note that the integrand defines a gauge invariant function $f:\mathfrak{u}(n)\to\mathbb{R}$. It can therefore be written using the Weyl integral formula as

$$\int_{\mathfrak{u}(n)} [\mathrm{d}\phi]\ f(\phi) = \frac{\mathrm{vol}(U(n))}{n!\,(2\pi)^n} \int_{\mathbb{R}^n} [\mathrm{d}s]\ \Delta(s)^2\ f(s)\ , \tag{6.12}$$

where the Vandermonde determinant is the Weyl determinant for $U(n)$ arising as the jacobian for the diagonalization of hermitian matrices on the left-hand side of (6.12). From (6.10)–(6.12) we obtain

$$\begin{aligned} Z_0' &= \frac{1}{\mathrm{vol}(S)} \int_{\mathfrak{s}} \left[\frac{\mathrm{d}\phi}{2\pi}\right]\ \mathrm{e}^{-\,\mathrm{i}\ \mathrm{Tr}(C_0\,\phi)-\frac{g'}{2}\ \mathrm{Tr}(\phi^2)} \\ &= \frac{1}{n!}\,\frac{1}{(2\pi)^{n^2}} \int_{\mathbb{R}^n} \left[\frac{\mathrm{d}s}{2\pi}\right]\ \Delta(s)^2\ \mathrm{e}^{-\,\mathrm{i}\,\frac{N}{2}\sum_i s_i-\frac{g}{4}\sum_i s_i^2} \end{aligned} \tag{6.13}$$

where we used $\mathrm{vol}(S)=N^{N^2/2}\ \mathrm{vol}(U(n))$ with respect to the Cartan-Killing metric on $\mathfrak{s}$, since $S=U(n)\otimes\mathbb{1}_N$. Applying the integral identity

$$\begin{aligned} &\int_{\mathbb{R}^n} [\mathrm{d}s]\ \Delta(s)^2\ \mathrm{e}^{-\,\mathrm{i}\,\frac{N}{2}\sum_i s_i+\frac{\mathrm{i}}{4}\sum_i m_i\,s_i-\frac{g}{4}\sum_i s_i^2} \\ &\qquad\qquad = \mathrm{e}^{-\frac{n\,N^2-m\,N}{4g}} \int_{\mathbb{R}^n} [\mathrm{d}s]\ \Delta(s)^2\ \mathrm{e}^{\frac{\mathrm{i}}{4}\sum_i m_i\,s_i-\frac{g}{4}\sum_i s_i^2} \end{aligned} \tag{6.14}$$

where $m=\sum_i m_i$ allows us to finally write the partition function as in (6.2).

7. Nonabelian localization at maximally irreducible saddle points

We now turn to the opposite extreme and look at the local contribution to the partition function (4.50) from a generic maximally non-degenerate critical surface. We denote this gauge orbit by

$$\mathcal{O}_{\max}(\vec{n}) := \mathcal{C}_{(n_1,1),\dots,(n_n,1)} = \{g\,C\,g^{-1}\ |\ g\in U(n\,N-\mathsf{c}_1)\} \cong U(n\,N-\mathsf{c}_1)/U(1)^n \tag{7.1}$$

and assume that the integers $n_1>n_2>\cdots>n_n$ are explicitly specified. Here we allow also $\mathsf{c}_1\neq 0$ which describes sectors with non-vanishing $U(1)$ monopole number.

7.1. Statement of result

Theorem 2. *The contribution to the quantum partition function for $U(n)$ Yang-Mills theory on S^2_N from a maximally non-degenerate moduli space $\mathcal{O}_{\max}(\vec{n})$ is given by*

$$Z_{\max} = \frac{(-1)^{n\,(n-1)/2}\ \mathrm{e}^{n\,N^2/4g}}{(2\pi)^{n^2+n}} \int_{\mathbb{R}^n} [\mathrm{d}s]\ \Delta(s)^2\ \mathrm{e}^{-\frac{\mathrm{i}}{2}\,N\,\sum_i s_i - \frac{g}{4}\,\sum_i \frac{n_i}{N}\,s_i^2}\ . \tag{7.2}$$

Setting $\tilde{s}_i := \sqrt{n_i/N}\, s_i$ in (7.2), we get

$$Z_{\max} = \frac{(-1)^{n\,(n-1)/2}}{(2\pi)^{n^2+n}}\ \frac{N^{n/2}\ \mathrm{e}^{n\,N^2/4g}}{\prod\limits_{k=1}^{n}\sqrt{n_k}} \int_{\mathbb{R}^n} [\mathrm{d}\tilde{s}\,] \prod_{k>l} \Big(\sqrt{\tfrac{N}{n_k}}\,\tilde{s}_k - \sqrt{\tfrac{N}{n_l}}\,\tilde{s}_l\Big)^2\ \mathrm{e}^{-\frac{\mathrm{i}}{2}\,\sum_i \sqrt{\frac{N^3}{n_i}}\,\tilde{s}_i - \frac{g}{4}\,\sum_i \tilde{s}_i^2}\ . \tag{7.3}$$

Completing the square of the gaussian function of $\tilde{s}_i$ in (7.3) identifies the Boltzmann weight of the action (4.30) on the non-degenerate solution space $\mathcal{C}_{(n_1,1),\dots,(n_n,1)}$. In the large N limit, we substitute (4.25) with $\tilde{s}_i \approx (1 + \frac{m_i}{2N})\, s_i$. Neglecting terms of order $\frac{1}{N}$ then reduces (7.3) to

$$Z_{\max} \approx \pm\, \frac{\mathrm{e}^{n\,N^2/4g}}{(2\pi)^{n^2+n}} \int_{\mathbb{R}^n} [\mathrm{d}s]\ \Delta(s)^2\ \mathrm{e}^{-\frac{\mathrm{i}}{2}\,N\,\sum_i s_i}\ \mathrm{e}^{\frac{\mathrm{i}}{4}\,\sum_i m_i\,s_i - \frac{g}{4}\,\sum_i s_i^2}\ , \tag{7.4}$$

and an application of the integral identity (6.14) leads to the result

$$Z_{\max} \approx \pm\, \frac{1}{(2\pi)^{n^2+n}} \int_{\mathbb{R}^n} [\mathrm{d}s]\ \Delta(s)^2\ \mathrm{e}^{\frac{\mathrm{i}}{4}\,\sum_i m_i\,s_i - \frac{g}{4}\,\sum_i s_i^2}\ . \tag{7.5}$$

This can easily be generalized [14] to non-trivial $U(1)$ monopole number, or Chern class $\mathsf{c}_1 = m = \sum_i m_i$. The form (7.5) coincides with the classical result [19] for the contribution to the $U(n)$ sphere partition function from the Yang-Mills instanton on S^2 specified by the configuration of magnetic monopole charges $m_1, \dots, m_n \in \mathbb{Z}$. In particular, using the standard manipulation of [19] one can change integration variables in (7.5) to identify the anticipated Boltzmann weight of the action (4.28).

7.2. Proof of Theorem 2

We want to compute the integral $Z'_{\max}$ in (4.50) over a local neighbourhood $\mathcal{N}_{\max}$ of $\mathcal{O}_{\max}(\vec{n})$, which is independent of t in the large t limit. This is similar in spirit but technically more involved than for the vacuum surface. We first need to find a suitable basis for the tangent space $T_C\mathcal{O}$ at the irreducible critical point C, using the splitting (5.17). The definition of the basis $J_i, \tilde{J}_i$ introduced in (6.4) naturally extends to include the non-trivial subspaces E_0, E_1 in this case with

$$J_i = \mathcal{J}(g'_i\,)\ , \quad \tilde{J}_j = \mathcal{J}^2(g'_j)\ , \quad H_i = \mathcal{J}(h'_i\,)\ \in\ \mathcal{J}(\mathfrak{h}) = E_0 \qquad \text{and} \qquad K_i \in E_1\ , \tag{7.6}$$

for g'_i and h'_i an orthonormal basis of $\mathfrak{g} \ominus \mathfrak{h} \ominus \mathfrak{s}$ and of $\mathfrak{h} \ominus \mathfrak{s}$, respectively. We define again $\langle \alpha, \mathcal{J}(g'_i\,)\rangle = f_i$. The elements K_i are assumed to form an orthonormal basis of E_1, orthogonal to $\mathcal{J}(\mathfrak{g}) \oplus \mathcal{J}^2(\mathfrak{g})$.

E_0 and E_1 are naturally complex vector spaces, whose generators are embedded into the tangent space decomposition (5.17) as

$$K_i = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & X_i & 0 \\ \hline 0 & X_i^\dagger & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{7.7}$$

and similarly for H_i. The complex structure is given by the map $\mathcal{J}$, which amounts to multiplying X_i by i. We accordingly take the real basis K_i to be ordered as $\{K_i\} = \{(\tilde{K}_i, \mathcal{J}(\tilde{K}_i))\}$, and similarly for H_i. As matrices, all of the generators H_i, K_j are hermitian. The corresponding dual one-forms β^i, γ^i are defined as usual by

$$\langle \beta^i \,, H_j \rangle = \delta^i{}_j \qquad \text{and} \qquad \langle \gamma^i \,, K_j \rangle = \delta^i{}_j \tag{7.8}$$

with all other pairings equal to 0. One can show [14] that

$$\mathrm{d}\alpha = \mathrm{d}f_i \wedge \lambda^i + \tfrac{1}{2}\, A_{ij}\ \gamma^i \wedge \gamma^j + O_f \tag{7.9}$$

where O_f denotes contributions which vanish on-shell, and

$$A_{ij} = 2\,\mathrm{i}\ \mathrm{Tr}\left(K_i\ \mathrm{ad}_{C_0}(K_j)\right) \tag{7.10}$$

is an antisymmetric matrix. One then has

$$\frac{(\mathrm{d}\alpha)^{d-d_0}}{(d-d_0)!} = \mathrm{pfaff}(A)\ \Big(\bigwedge_{i=1}^{2d_1}\gamma^i\Big)\ \wedge\ \Big(\bigwedge_{j=1}^{d-d_0-d_1}\mathrm{d}f_j \wedge \lambda^j\Big) + O_f \tag{7.11}$$

where d_0 (resp. d_1) is the complex dimension of the vector space E_0 (resp. E_1), and

$$\mathrm{pfaff}(A) = \epsilon^{i_1\cdots i_{2d_1}}\ A_{i_1 i_2}\cdots A_{i_{2d_1-1} i_{2d_1}} \tag{7.12}$$

is the pfaffian of the antisymmetric matrix $A = (A_{ij})$.

To proceed with the localization, we need to find the local geometry and define its symplectic model. The G-equivariant tubular neighbourhood $\mathcal{N}_{\max}$ of $\mathcal{O}_{\max}(\vec{n})$ has an equivariant retraction (by a local equivariant symplectomorphism) onto the local symplectic model $\mathcal{F}_{\max}$, defined to be an equivariant symplectic vector bundle over $\mathcal{O}_{\max}(\vec{n})$ with fibre $\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s})\oplus E_1$ which is a sub-bundle of the tangent bundle $T\mathcal{O}$ restricted to $\mathcal{O}_{\max}(\vec{n})$. Thus the tangent space to $\mathcal{F}_{\max}$ is given by

$$T_C\mathcal{O}_{\max}(\vec{n}) \oplus \mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s}) \oplus E_1 \ \cong\ E_0 \oplus \mathcal{J}(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s}) \oplus \mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s}) \oplus E_1 \ =\ T_C\mathcal{O}\ , \tag{7.13}$$

and the symplectic two-form on $\mathcal{F}_{\max}$ is simply ω. In physical terms, the gauge fields are split along the moduli space $\mathcal{O}_{\max}(\vec{n})$, plus infinitesimal non-gauge variations belonging to $\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s})$ along with unstable modes in the subspace E_1. Due to the presence of the localization form α in the action, we can restrict ourselves to this model $\mathcal{F}_{\max}$ replacing $\mathcal{N}_{\max}$. Identically to the case of the vacuum surface in the previous section, the canonical symplectic integral over $\mathfrak{g}\times\mathcal{N}_{\max}$ will in this way reduce to an integral over $\mathfrak{s}\times\mathcal{O}_{\max}(\vec{n})$.

We may now proceed to calculate

$$\begin{aligned}
Z'_{\max} &= \frac{1}{\mathrm{vol}(G)}\int_{\mathfrak{g}\times\mathcal{N}_{\max}}\Big[\frac{\mathrm{d}\phi}{2\pi}\Big]\ \exp\Big(\omega + t\,(\mathrm{d}\alpha - \mathrm{i}\,\langle\alpha, V_\phi\rangle) - \mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \tfrac{g'}{2}\ \mathrm{Tr}\,(\phi^2)\Big) \\
&= \frac{1}{\mathrm{vol}(G)}\int_{\mathfrak{g}\times\mathcal{O}_{\max}(\vec{n})\times\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s})\times E_1}\Big[\frac{\mathrm{d}\phi}{2\pi}\Big]\ \frac{(t\ \mathrm{d}\alpha)^{d-d_0}}{(d-d_0)!}\wedge\frac{\omega^{d_0}}{d_0!} \\
&\qquad\qquad \times\ \mathrm{e}^{-\mathrm{i}\,t\,\langle\alpha,V_\phi\rangle - \mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \frac{g'}{2}\ \mathrm{Tr}(\phi^2)} \\
&= \frac{1}{\mathrm{vol}(G)}\int_{(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s})\oplus\mathfrak{h}\oplus\mathfrak{s}}\Big[\frac{\mathrm{d}\phi}{2\pi}\Big]\ \mathrm{pfaff}(A) \\
&\qquad\qquad \times\int_{\mathcal{O}_{\max}(\vec{n})\times\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s})\times E_1} t^{d-d_0}\ \Big(\bigwedge_{i=1}^{2d_1}\gamma^i\Big)\ \wedge\ \Big(\bigwedge_{j=1}^{d-d_0-d_1}\mathrm{d}f_j\wedge\lambda^j\Big)\ \wedge\ \frac{\omega^{d_0}}{d_0!} \\
&\qquad\qquad \times\ \mathrm{e}^{-\mathrm{i}\,t\,(N\,f_i\,\phi^i + \langle\alpha,V_{\phi'}\rangle) - \mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \frac{g'}{2}\ \mathrm{Tr}(\phi^2)}
\end{aligned} \tag{7.14}$$

with $\phi' \in \mathfrak{h} \oplus \mathfrak{s}$. In the second line we have used the fact that $\mathrm{d}\alpha$ vanishes when evaluated on the subspace E_0, and therefore we need d_0 powers of ω to yield a non-trivial volume form. Then $(t\ \mathrm{d}\alpha)^{d-d_0} \wedge \omega^{d_0}$ is the only term which survives in the large t limit. We will modify this below by adding a second localization form α' in order to write the localization integral in the generic form (4.50) without the symplectic two-form ω.

We can now evaluate the integrals in (7.14) over f_i in the fibre $\mathcal{J}^2(\mathfrak{g} \ominus \mathfrak{h} \ominus \mathfrak{s})$ and $\phi^i \in \mathfrak{g} \ominus \mathfrak{h} \ominus \mathfrak{s}$ as in the previous section, which localizes for $t \to \infty$ to an integral over the subspace E_1 and the gauge orbit $\mathcal{O}_{\max}(\vec{n})$ given by

$$Z'_{\max} = \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{h}\oplus\mathfrak{s}} \left[\frac{\mathrm{d}\phi}{2\pi}\right] \frac{\mathrm{pfaff}(A)}{N^{d-d_0-d_1}} \int_{\mathcal{O}_{\max}(\vec{n})\times E_1} t^{d_1} \Big(\bigwedge_{i=1}^{2d_1} \gamma^i\Big) \wedge \Big(\bigwedge_{j=1}^{d-d_0-d_1} \lambda^j\Big) \wedge \frac{\omega^{d_0}}{d_0!} \times \mathrm{e}^{-\mathrm{i}\, t\, \langle\alpha, V_\phi\rangle - \mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \frac{g'}{2}\ \mathrm{Tr}(\phi^2)} \ . \tag{7.15}$$

The gauge invariant volume form for the integration domain whose tangent space is E_0 is given by the symplectic volume form $\omega^{d_0}/d_0!$, since $\mathrm{d}\alpha$ vanishes on E_0, but this will be modified below. It remains to compute the integral over E_1. Upon evaluating $\langle\alpha, V_\phi\rangle$ at second order on E_1, i.e. away from the critical surface, we will find below that this pairing becomes a quadratic form which leads to a localization through a gaussian integral. However, to evaluate it explicitly it is easier to first localize the integral over E_0, which presently is a complicated non-gaussian integral which does not admit a gaussian approximation at $t \to \infty$ and is difficult to evaluate in a closed analytic form. But this can be done by adapting a trick taken from [18], which amounts to adding a further suitable localization one-form α', or equivalently a cohomologically trivial form $Q\alpha'$, to the action in (4.50). Indeed, we may compute $Z'_{\max}$ using any other invariant form α' which is homotopic to α on the open neighbourhood $\mathcal{N}_{\max}$. The one-form α' need only be non-vanishing on $E_0 \subset \mathcal{N}_{\max}$, as the other integrals can be directly carried out.

In order to evaluate the integrals over E_0 and $\mathfrak{h}$, following [18] we introduce an additional localization term $\exp(t\ Q\alpha')$ in the partition function with

$$\alpha' := -\tfrac{2}{N}\, \mathcal{J}\ \mathrm{d}\,\mathrm{Tr}(C\,\phi)\Big|_{E_0} \ . \tag{7.16}$$

The projection onto E_0 is equivalent to projecting $\phi \in \mathfrak{g}$ onto $\mathfrak{h}$. This one-form is equivariant on-shell, and it can be extended to the G-equivariant tubular neighbourhood $\mathcal{N}_{\max}$ of the critical surface $\mathcal{O}_{\max}(\vec{n})$ as follows. On the tangent space $\mathcal{J}(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s}) \oplus E_0$ of $T\mathcal{O}_{\max}(\vec{n})$ in (7.13) there is an equivariant projection onto the subspace E_0. In this way α' is properly defined on the local model, and can hence be extended to $\mathcal{N}_{\max}$. One could also define $\alpha' = \frac{2\,\mathrm{i}}{N}\,\chi\,\mathcal{J}\ \mathrm{d}\,\mathrm{Tr}(C\,\phi)|_{E_0}$ using a smooth G-invariant cutoff function χ with support near the given saddle-point and $\chi = 1$ in the tubular neighbourhood, which is globally well-defined over $\mathcal{N}_{\max}$ as an equivariant differential form. Note that $t_1\,\alpha + t_2\,\alpha'$ vanishes only on the original critical points for any $t_1, t_2 \in \mathbb{R}$ with $t_1 \neq 0$, and no new ones are introduced. Then our previous computation (4.47) would essentially go through, since α' vanishes on $\mathcal{J}(\mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s})$ and there are no critical points where $\mathrm{d}\chi \neq 0$. It is therefore just as good a localization form to use as α is. It follows that the modification of the canonical symplectic integral over $\mathcal{N}_{\max}$ given by

$$Z'_{\max} = \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{N}_{\max}} \left[\frac{\mathrm{d}\phi}{2\pi}\right]\ \exp\Big(\omega + t_1\ Q\alpha + t_2\ Q\alpha' - \mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \tfrac{g'}{2}\ \mathrm{Tr}\,(\phi^2)\Big) \tag{7.17}$$

is independent of both $t_1, t_2 \in \mathbb{R}$. Then α' will localize the integral over $\mathfrak{h} \subset \mathfrak{g}$ as well as the integral over the unstable modes in E_1, without the need to expand $\langle\alpha, V_\phi\rangle$ to higher order.

Let us first integrate over $\mathfrak{h}$. One can show [14] that the new localization form α' satisfies

$$\langle\alpha', V_{h_i}\rangle = 2\,\mathrm{Tr}\,(H_i\,\mathcal{J}(\phi)) \ , \tag{7.18}$$

with h_i a basis of $\mathfrak{h}$. This produces a gaussian integral localizing $\mathfrak{h}$ to the gauge stabilizer algebra $\mathfrak{s} \cong \mathfrak{u}(1)^n$. Then

$$\mathrm{d}\alpha' = \tfrac{2\,\mathrm{i}}{N^2}\,\tilde{A}_{ij}\ \beta^i \wedge \beta^j \qquad \text{and} \qquad \frac{(\mathrm{d}\alpha'\,)^{d_0}}{d_0!} = \big(\tfrac{4\,\mathrm{i}}{N^2}\big)^{d_0}\ \ \mathrm{pfaff}(\tilde{A}\,)\ \bigwedge_{i=1}^{2d_0} \beta^i\ , \tag{7.19}$$

where $\tilde{A}_{ij} := \mathrm{Tr}(H_i\,[s,H_j])$ is an antisymmetric matrix and we have restricted to $\phi = s \in \mathfrak{s}$ using the localization (see (7.22) below). Using the explicit description of the local geometry given in Section 5, one finds

$$\mathrm{pfaff}(\tilde{A}\,) = (-\,\mathrm{i}\,)^{d_0}\sqrt{\det(M)}\ \prod_{k>l}(s_k - s_l)^{|n_k-n_l|+1} \tag{7.20}$$

where $M_{ij} := 2\,\mathrm{Tr}(H_i\,H_j)$ is a symmetric matrix. We can now evaluate the localization integral

$$\int_{\mathfrak{h}}\Big[\frac{\mathrm{d}\phi}{2\pi}\Big]\ t_2^{d_0}\ \frac{(\mathrm{d}\alpha'\,)^{d_0}}{d_0!}\ \mathrm{e}^{-\,\mathrm{i}\,t_2\,\langle\alpha',V_\phi\rangle} = \big(\tfrac{4\,\mathrm{i}}{N^2}\big)^{d_0}\ \int_{\mathfrak{h}}\Big[\frac{\mathrm{d}\phi}{2\pi}\Big]\ t_2^{d_0}\ \mathrm{pfaff}(\tilde{A}\,)\ \mathrm{e}^{-2\,\mathrm{i}\,t_2\,\phi^i\,M_{ij}\,\phi^j}\ \bigwedge_{i=1}^{2d_0}\beta^i \tag{7.21}$$

where $\phi = \phi^i\,h_i$. The oscillatory gaussian integral is defined by analytic continuation $t_2 \to t_2 - \mathrm{i}\,\varepsilon$ for a small positive parameter ε, which we are free to do as the partition function is formally independent of t_2. With this continuation understood and a suitable orientation of the vector space $\mathfrak{h}$, we readily compute

$$\begin{aligned}\int_{\mathfrak{h}}\Big[\frac{\mathrm{d}\phi}{2\pi}\Big]\ t_2^{d_0}\ \frac{(\mathrm{d}\alpha'\,)^{d_0}}{d_0!}\ \mathrm{e}^{-\,\mathrm{i}\,t_2\,\langle\alpha',V_\phi\rangle} &= \big(\tfrac{4\,\mathrm{i}}{N^2}\big)^{d_0}\ \big(\tfrac{1}{2\pi}\big)^{2d_0}\ \big(-\tfrac{\pi}{2\,\mathrm{i}}\big)^{d_0}\ \frac{\mathrm{pfaff}(\tilde{A}\,)}{\sqrt{\det(M)}}\ \bigwedge_{i=1}^{2d_0}\beta^i \\ &= \frac{\mathrm{i}^{\,d_0}}{(2\pi\,N^2)^{d_0}}\ \prod_{k>l}(s_k-s_l)^{|n_k-n_l|+1}\ \bigwedge_{i=1}^{2d_0}\beta^i\ .\end{aligned} \tag{7.22}$$

This integral thus produces a measure on $\mathfrak{s}$ which we will use below to perform the remaining integral over the stabilizer.

Now that the ϕ-integration in (7.15) is localized onto $\mathfrak{s}$, we can proceed to evaluate the integral over E_1. This space has a basis K_i as introduced in (7.7). We need to evaluate $\langle\alpha,V_s\rangle$ for $s \in \mathfrak{s}$ up to second order in the fluctuations about the critical point in E_1, which is non-tangential to the gauge orbit $\mathcal{O}_{\max}(\vec{n})$. For this, we introduce real linear coordinates x^i, y^i, $i = 1,\dots,d_1$ on E_1 such that a generic vector $V_\Psi \in E_1$ is parametrized as $V_\Psi = (x^i\,K_i\,,\,y^i\,\mathcal{J}(K_i))$. Then $\gamma^i = \mathrm{d}x^i$ and $\gamma^{i+d_1} = \mathrm{d}y^i$ for $i = 1,\dots,d_1$. We can choose coordinates on $T_C\mathcal{O}$ such that $G_{ij} = 2\ \mathrm{Tr}\,(X_i\,X_j^\dagger)$ is diagonal. One then finds

$$\langle\alpha,V_s\rangle = -\,\mathrm{Tr}\,(\mathrm{ad}_s(V_\Psi)\ \mathrm{ad}_{C_0}(V_\Psi)) = (x^i\,,\,y^i)\ \tilde{M}_{ij}(s)\ \begin{pmatrix} x^j \\ y^j\end{pmatrix} \tag{7.23}$$

to second order, where

$$\tilde{M}_{ij}(s) \ = \ (s_k - s_l)\,c_{kl}\ \begin{pmatrix} G & 0 \\ 0 & G\end{pmatrix}_{ij} \tag{7.24}$$

is a symmetric matrix and

$$c_{kl} = \frac{N}{2}\ \frac{n_l - n_k}{n_k\,n_l}\ . \tag{7.25}$$

One finds similarly

$$\mathrm{pfaff}(A) = 2^{d_1}\ \sqrt{\det\big(\tilde{M}(s)\big)}\ \prod_{k>l}(s_k - s_l)^{1-|n_k-n_l|}\ . \tag{7.26}$$

These pfaffians are the typical representatives of fluctuations in equivariant localization [16], as discussed in Section 2. Using the analytic continuation $t_1 \to t_1 - \mathrm{i}\,\varepsilon$ and a suitable orientation of E_1 as before, we can now evaluate the oscillatory gaussian integral

$$\int_{E_1}\ \prod_{i=1}^{d_1} \mathrm{d}x^i\ \mathrm{d}y^i\ t_1^{d_1}\ \mathrm{e}^{-\,\mathrm{i}\,t_1\,\langle\alpha,V_s\rangle} = \Big(\frac{\pi}{\mathrm{i}}\Big)^{d_1}\ \frac{1}{\sqrt{\det\big(\tilde{M}(s)\big)}}\ . \tag{7.27}$$

Finally, putting the results (7.15), (7.22), (7.26) and (7.27) together, we may evaluate the large t_1, t_2 limit of the desired symplectic integral (7.17) to obtain

$$\begin{aligned}
Z'_{\max} &= \frac{1}{\mathrm{vol}(G)}\int_{\mathfrak{g}\times\mathcal{F}_{\max}}\Big[\frac{\mathrm{d}\phi}{2\pi}\Big]\ \exp\Big(\mathrm{d}(t_1\,\alpha + t_2\,\alpha'\,) -\ \mathrm{i}\,\langle t_1\,\alpha + t_2\,\alpha', V_\phi\rangle\Big) \\
&\qquad \times\ \mathrm{e}^{-\,\mathrm{i}\ \mathrm{Tr}(C_0\,\phi) - \frac{g'}{2}\ \mathrm{Tr}(\phi^2)} \\
&= \frac{1}{\mathrm{vol}(G)}\ \Big(\frac{\pi}{\mathrm{i}}\Big)^{d_1}\ \frac{\mathrm{i}^{\,d_0}}{(2\pi\,N^2)^{d_0}}\ \int_{\mathfrak{s}}\Big[\frac{\mathrm{d}s}{2\pi}\Big]\ \prod_{k>l}(s_k - s_l)^{|n_k-n_l|+1}\ \frac{\mathrm{pfaff}(A)}{\sqrt{\det\big(\tilde{M}(s)\big)}} \\
&\qquad \times\ \frac{1}{N^{d-d_0-d_1}}\int_{\mathcal{O}_{\max}(\vec{n})}\Big(\bigwedge_{j=1}^{d-d_0-d_1}\lambda^j\Big)\ \wedge\ \Big(\bigwedge_{i=1}^{2d_0}\beta^i\Big)\ \mathrm{e}^{-\,\mathrm{i}\ \mathrm{Tr}(C_0\,s) - \frac{g'}{2}\ \mathrm{Tr}(s^2)} \\
&= \frac{1}{\mathrm{vol}(G)}\ \frac{\mathrm{i}^{\,d_0-d_1}}{(2\pi)^{d_0-d_1}}\ \prod_{k=1}^{n}\sqrt{n_k}\ \int_{\mathbb{R}^n}\Big[\frac{\mathrm{d}s}{2\pi}\Big]\ \Delta(s)^2\ \mathrm{e}^{-\,\mathrm{i}\ \mathrm{Tr}(C_0\,s) - \frac{g'}{2}\ \mathrm{Tr}(s^2)} \\
&\qquad \times\ \frac{1}{N^{d+d_0-d_1}}\int_{\mathcal{O}_{\max}(\vec{n})}\Big(\bigwedge_{j=1}^{d-d_0-d_1}\lambda^j\Big)\ \wedge\ \Big(\bigwedge_{i=1}^{2d_0}\beta^i\Big)
\end{aligned} \tag{7.28}$$

where we have transformed the integration over $\phi = s = \mathrm{diag}(s_1\ \mathbb{1}_{n_1}, \dots, s_n\ \mathbb{1}_{n_n}) \in \mathfrak{s}$ to an integral over $s = (s_1, \dots, s_n) \in \mathbb{R}^n$. We can carry out the integral over the moduli space $\mathcal{O}_{\max}(\vec{n})$ by observing again

$$\frac{1}{N^{d+d_0-d_1}}\int_{\mathcal{O}_{\max}(\vec{n})}\Big(\bigwedge_{j=1}^{d-d_0-d_1}\lambda^j\Big)\ \wedge\ \Big(\bigwedge_{i=1}^{2d_0}\beta^i\Big) = \int_{G/S}\ \bigwedge_{j=1}^{d+d_0-d_1}\eta^j = \frac{\mathrm{vol}(G)}{\mathrm{vol}(S)}\ , \tag{7.29}$$

where $\mathcal{J}^*(\lambda^i) = \eta^i$ are left-invariant one-forms on the gauge group G. Note that (7.29) includes the integral over E_0, and $\dim_{\mathbb{R}}(\mathfrak{g}\ominus\mathfrak{s}) = d + d_0 - d_1$. We also have $\mathrm{vol}(S) = \prod_k\ 2\pi\ \sqrt{n_k}$ in our metric on $\mathfrak{s}$, since $S = \prod_k\ U(1)\otimes\mathbb{1}_{n_k}$, and $C_0(n_i) = \frac{N}{2n_i}\ \mathbb{1}_{n_i}$. Using furthermore $d_0 - d_1 = n^2 - n$ which is an even integer, we may then bring (7.28) into the form

$$Z'_{\max} = \frac{\mathrm{i}^{\,n^2-n}}{(2\pi)^{n^2+n}}\ \int_{\mathbb{R}^n}[\mathrm{d}s]\ \Delta(s)^2\ \mathrm{e}^{-\,\mathrm{i}\ \mathrm{Tr}(C_0\,s) - \frac{g'}{2}\ \mathrm{Tr}(s^2)} \tag{7.30}$$

which immediately leads to (7.2).

Acknowledgments
We would like to thank the organizers of the Orsay meeting for the invitation and hospitality in a stimulating atmosphere. The work of H.S. is supported by the FWF project P18657. R.J.S. was supported in part by the EU-RTN Network Grant MRTN-CT-2004-005104.

[1] M.R. Douglas and N.A. Nekrasov, *Rev. Mod. Phys.* **73** (2001) 977.
[2] R.J. Szabo, *Phys. Rept.* **378** (2003) 207.
[3] H. Steinacker, arXiv:0708.2426 [hep-th].
[4] R. J. Szabo, Class. Quant. Grav. **23** (2006) R199 [arXiv:hep-th/0606233].
[5] J. Madore, *Class. Quant. Grav.* **9** (1992) 69.
[6] U. Carow-Watamura and S. Watamura, *Commun. Math. Phys.* **212** (2000) 395.
[7] H. Grosse, C. Klimcik and P. Presnajder, *Int. J. Theor. Phys.* **35** (1996) 231.
[8] C. Klimcik, *Commun. Math. Phys.* **199** (1998) 257.
[9] H. Steinacker, *Nucl. Phys. B* **679** (2004) 66.
[10] H. Grosse and A. Strohmaier, *Lett. Math. Phys.* **48** (1999) 163.
[11] G. Alexanian, A.P. Balachandran, G. Immirzi and B. Ydri, *J. Geom. Phys.* **42** (2002) 28.
[12] H. Grosse and H. Steinacker, *Nucl. Phys. B* **707** (2005) 145.
[13] W. Behr, F. Meyer and H. Steinacker, *J. High Energy Phys.* **0507** (2005) 040.
[14] H. Steinacker and R.J. Szabo, "Localization for Yang-Mills theory on the fuzzy sphere," *Preprint* arXiv:hep-th/0701041.
[15] L.D. Paniak and R.J. Szabo, *Commun. Math. Phys.* **243** (2003) 343.
[16] R.J. Szabo, *Lect. Notes Phys.* **M63** (2000) 1; "Equivariant Localization of Path Integrals", *Preprint* arXiv:hep-th/9608068.
[17] E. Witten, *J. Geom. Phys.* **9** (1992) 303.
[18] C. Beasley and E. Witten, *J. Diff. Geom.* **70** (2005) 183
[19] J.A. Minahan and A.P. Polychronakos, *Nucl. Phys. B* **422** (1994) 172.
[20] D.J. Gross and A. Matytsin, *Nucl. Phys. B* **429** (1994) 50.